

# Poisson structure and invariant manifolds on Lie groups

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## Abstract

For a discrete mechanical system on a Lie group  $G$  determined by a (reduced) Lagrangian  $\ell$  we define a Poisson structure via the pull-back of the Lie-Poisson structure on  $\mathfrak{g}^*$  by the corresponding Legendre transform. The main result shown in this paper is that this structure coincides with the reduction under the symmetry group  $G$  of the canonical discrete Lagrange 2-form  $\omega_L$  on  $G \times G$ . Its symplectic leaves then become dynamically invariant manifolds for the reduced discrete system.

## 1 Introduction

**Background.** For systems on finite dimensional Lie groups  $G$  with Lagrangians  $L : TG \rightarrow \mathbb{R}$  that are  $G$ -invariant, discrete analogues of Euler-Poincaré and Lie-Poisson reduction theory (see, for example, Marsden and Ratiu [MR 99]) were developed in [MPeS 98]. The resulting discrete equations provide “reduced” numerical algorithms which manifestly preserve the symplectic structure. The manifold  $G \times G$  is used as the discrete approximation of  $TG$ , and a discrete Lagrangian  $\mathbb{L} : G \times G \rightarrow \mathbb{R}$  is constructed in such a way that the  $G$ -invariance property is preserved. Reduction by  $G$  results in a new “variational” principle for the reduced Lagrangian  $\ell : G \rightarrow \mathbb{R}$ , which then determines the discrete Euler-Poincaré (DEP) equations. Reconstruction of these equations is consistent with the usual Veselov discrete Euler-Lagrange equations developed in [WM 97, MPS 98], which are naturally symplectic-momentum algorithms. Furthermore, the solution of the DEP algorithm leads directly to a discrete Lie-Poisson (DLP) algorithm. For the reader’s benefit we summarize main results of the discrete reduction in the Appendix.

**Motivation.** Discretization of an Euler-Poincaré system on  $TG$  results in a system on  $G \times G$  defined by a Lagrangian  $\mathbb{L}$ . If it is regular, the Legendre transformations  $F\mathbb{L}$  define a symplectic form (and, hence, a Poisson structure) on  $\Delta \subset G \times G$  via the pull-back of the canonical form from  $T^*G$ . Then, general Poisson reduction applied to these discrete settings defines a Poisson structure on the reduced space  $\mathcal{U} := \pi_d(\Delta) \subset G$ . This approach was adopted in Theorem 2.2 of [MPeS 98].

Alternatively, without appealing to the reduction procedure, a Poisson structure on a Lie group can be defined using ideas of Weinstein [W 96] on Lagrangian mechanics on groupoids and their algebroids. The key idea can be summarized in the following statements. A smooth function on a groupoid defines a natural (Legendre type) transformation between the groupoid and the dual of its algebroid. This transformation can be used to pull back a canonical Poisson structure from the dual of the algebroid, provided the regularity conditions are satisfied.

## 2 Dynamics on groupoids and algebroids

We briefly summarize results from Weinstein [W 96] and refer the reader to the original paper for details of proofs and definitions. Let  $\Gamma$  be a groupoid over a set  $M$ , with  $\alpha, \beta : \Gamma \rightarrow M$  being its source and

target maps, with a multiplication map  $m : \Gamma_2 \rightarrow \Gamma$ , where  $\Gamma_2 \equiv \{(g, h) \in \Gamma \times \Gamma \mid \beta(g) = \alpha(h)\}$ . Denote its corresponding algebroid by  $\mathcal{A}$ .

The Lie groupoids relevant to our exposition are the Cartesian product  $G \times G$  of a Lie group  $G$ , with multiplication  $(g, h)(h, k) = (g, k)$ , and the group  $G$  itself. The corresponding algebroids are the tangent bundle  $TG$  and the Lie algebra  $\mathfrak{g}$ , respectively. The dual bundle to a Lie algebroid carries a natural Poisson structure. This is the Poisson bracket associated to the canonical symplectic form on  $T^*G$  and the Lie-Poisson structure on  $\mathfrak{g}^*$ , respectively.

Lagrangian mechanics on a groupoid  $\Gamma$  is defined as follows. Let  $\mathcal{L}$  be a smooth, real-valued function on  $\Gamma$ ,  $\mathcal{L}_2$  the restriction to  $\Gamma_2$  of the function  $(g, h) \mapsto \mathcal{L}(g) + \mathcal{L}(h)$ .

**Definition 2.1.** *Let  $\Sigma_{\mathcal{L}} \subset \Gamma_2$  be the set of critical points of  $\mathcal{L}_2$  along the fibers of the multiplication map  $m$ ; i.e. the points in  $\Sigma_{\mathcal{L}}$  are stationary points of the function  $\mathcal{L}(g) + \mathcal{L}(h)$  when  $g$  and  $h$  are restricted to admissible pairs with the constraint that the product  $gh$  is fixed [W 96].*

A **solution of the Lagrange equations** for the Lagrangian  $\mathcal{L}$  is a sequence  $\dots, g_{-2}, g_{-1}, g_0, g_1, g_2, \dots$  of elements of  $\Gamma$ , defined on some “interval” in  $\mathbb{Z}$ , such that  $(g_j, g_{j+1}) \in \Sigma_{\mathcal{L}}$  for each  $j$ .

The Hamiltonian formalism for discrete Lagrangian systems is based on the fact that each Lagrangian submanifold of a symplectic groupoid determines a Poisson automorphism on the base Poisson manifold. Recall that the cotangent bundle  $T^*\Gamma$  is, in addition to being a symplectic manifold, a groupoid itself, the base being  $\mathcal{A}^*$ ; notice that both manifolds are naturally Poisson. The source and target mappings  $\tilde{\alpha}, \tilde{\beta} : T^*\Gamma \rightarrow \mathcal{A}^*$  are induced by  $\alpha$  and  $\beta$ .

**Definition 2.2.** *Given any smooth function  $\mathcal{L}$  on  $\Gamma$ , a Poisson map  $\Lambda_{\mathcal{L}}$  from  $\mathcal{A}^*$  to itself, which may be said to be generated by  $\mathcal{L}$  is defined by the Lagrangian submanifold  $d\mathcal{L}(\Gamma)$  (under a suitable hypothesis of nondegeneracy) [W 96].*

The appropriate “Legendre transformation”  $F\mathcal{L}$  in the groupoid context is given by  $\tilde{\alpha} \circ d\mathcal{L} : \Gamma \rightarrow \mathcal{L}^*$  or  $\tilde{\beta} \circ d\mathcal{L} : \Gamma \rightarrow \mathcal{L}^*$ , depending on whether we consider right or left invariance (through the definition of maps  $\tilde{\alpha}$  and  $\tilde{\beta}$ ). The transformation  $F\mathcal{L}$  relates the mapping on  $\Gamma$  defined by  $\Sigma_{\mathcal{L}}$  with the mapping  $\Lambda_{\mathcal{L}}$  on  $\mathcal{A}^*$ .  $F\mathcal{L}$  also pulls back the Poisson structure from  $\mathcal{A}^*$  to  $\Gamma$ , which, in general, is defined only *locally* on some neighborhood  $\mathcal{U} \subset \Gamma$ . In the context of a Lie group, this means that any regular function  $\ell : G \rightarrow \mathbb{R}$  defines a Poisson structure on  $\mathcal{U}$ . We shall address this issue in the next sections. The reader is referred to [W 96] for an application of the above ideas to the groupoid  $M \times M$  when the manifold  $M$  does not necessarily have group structure.

### 3 DEP as generators of Lie-Poisson Hamilton-Jacobi equations

A Lie group  $G$  is the simplest example of a groupoid with the base being just a point. Its algebroid is the corresponding Lie algebra  $\mathfrak{g}$ , with the dual being  $\mathfrak{g}^*$ . Consider left invariance and let a general function  $\mathcal{L}$  on the group be specified by the discrete reduced Lagrangian  $\ell : G \rightarrow \mathbb{R}$ . Then, the Legendre transform defined above is given by  $F\ell = L_g^* \circ d\ell : G \rightarrow \mathfrak{g}^*$ , where  $d\ell : G \rightarrow T^*G$ . Using these transformations we define  $\Pi_{k-1} \equiv F\ell(f_{kk-1}) = L_{f_{kk-1}}^* \circ d\ell(f_{kk-1})$ . Recall the DEP equation (4.4) for left-invariant systems :  $L_{f_{k+1k}}^* d\ell(f_{k+1k}) - R_{f_{kk-1}}^* d\ell(f_{kk-1}) = 0$ , where we have identified the notations  $\ell'$  and  $d\ell$ . The later equation can be rewritten as a system

$$\begin{cases} \Pi_k = L_f^* \circ d\ell(f), \\ \Pi_{k+1} = R_f^* \circ d\ell(f), \end{cases} \quad (3.1)$$

where the first equation is to be solved for  $f$  (which stands for  $f_{k+1k}$ ) which then is substituted into the second equation to compute  $\Pi_{k+1}$ .

This system is precisely the Lie-Poisson Hamilton-Jacobi system described in [GM 88] with the reduced discrete Lagrangian  $\ell$  playing the role of the generating function. This means that there is no need to find an approximate solution of the reduced Hamilton-Jacobi equation [GM 88]. Notice also that the DLP equation is a direct consequence of the system (3.1):  $\Pi_{k+1} = \text{Ad}_{f_{k+1k}}^* \cdot \Pi_k$ .

The following diagrams relate the dynamics on  $G$  and on  $\mathfrak{g}^*$ :

$$\begin{array}{ccc} G & \xrightarrow{\Sigma_\ell} & G \\ \downarrow F\ell & & \downarrow F\ell \\ \mathfrak{g}^* & \xrightarrow{\Lambda_\ell} & \mathfrak{g}^* \end{array} \quad \begin{array}{ccc} f_{kk-1} & \xrightarrow{\Sigma_\ell} & f_{k+1k} \\ \downarrow F\ell & & \downarrow F\ell \\ \Pi_{k-1} & \xrightarrow{\Lambda_\ell} & \Pi_k, \end{array} \quad (3.2)$$

where  $\Sigma_\ell$  and  $\Lambda_\ell$  are given in Definitions 2.1 and 2.2.

## 4 Some Advantages of Structure-preserving Integrators

As we mentioned above, the “Legendre transform”  $F\ell$  allows us to put a Poisson structure on the Lie group  $G$ , which, of course, depends on the discrete Lagrangian on  $G \times G$ , and hence on the original Lagrangian  $L$  on  $TG$  (if we consider this from the discrete reduction point of view). It follows that the reduction of the discrete Euler-Lagrange dynamics on  $G \times G$  is necessarily restricted to the symplectic leaves of this Poisson structure, so that these leaves are invariant manifolds, and correspond (under  $F\ell^*$ ) to the symplectic leaves (coadjoint orbits) of the continuous reduced system on  $\mathfrak{g}^*$ .

These ideas are the content of the following theorems. Here we state the theorems (for the case of right invariance) and only sketch their proofs. The reader is referred to [MPeS 99] for details. Analogous theorems hold for the case of left invariant systems.

**Theorem 4.1.** *Let  $L$  be a right invariant Lagrangian on  $TG$  and let  $\mathbb{L}$  be the Lagrangian of the corresponding discrete system on  $\mathcal{U} \subset G \times G$ . Assume that  $\mathbb{L}$  is regular, in the sense that the Legendre transformation  $F\mathbb{L} : \Delta \rightarrow F\mathbb{L}(\Delta) \subset T^*G$  is a local diffeomorphism, and let the quotient maps be given by*

$$\pi_d : G \times G \rightarrow (G \times G)/G \cong G \quad \text{and} \quad \pi : T^*G \rightarrow (T^*G)/G \cong \mathfrak{g}^*.$$

*Let  $\ell$  be the reduced Lagrangian on  $G$  defined by  $\mathbb{L} = \ell \circ \pi_d$ , and let  $F\ell : G \rightarrow \mathfrak{g}^*$  be the corresponding Legendre transform. Then the following diagram commutes:*

$$\begin{array}{ccc} G \times G & \xrightarrow{F\mathbb{L}} & T^*G \\ \downarrow \pi_d & & \downarrow \pi \\ G & \xrightarrow{F\ell} & \mathfrak{g}^*. \end{array} \quad (4.1)$$

*Idea of the proof.* Choosing appropriate coordinate systems on each space, we can rewrite this diagram as follows:

$$\begin{array}{ccc} (g_k, g_{k+1}) & \xrightarrow{F\mathbb{L}} & (g_k, p_k = \frac{\partial \mathbb{L}}{\partial g_k}) \\ \downarrow \pi_d & & \downarrow \pi \\ f = R_{g_{k+1}}^{-1} g_k & & \mu = R_{g_k}^* p_k, \end{array}$$

where  $f$  stands for  $f_{kk+1} = g_k g_{k+1}^{-1}$ . To close this diagram and to verify the arrow determined by  $F\ell$  compute the derivative of  $\mathbb{L}$  using the chain rule and use definitions of the partial derivative  $\partial f / \partial g_k$  and the Legendre transformation  $F\ell$ :

$$\mu = R_{g_k}^* p_k = R_{g_k}^* \frac{\partial(\ell \circ \pi)}{\partial g_k} = R_{g_k}^* \left( R_{g_{k+1}}^{-1} \frac{\partial \ell}{\partial f} \right) = R_f^* \frac{\partial \ell}{\partial f} = R_f^* \circ \ell'(f), \quad (4.2)$$

**Corollary 4.1.** *Reconstruction of the discrete Lie-Poisson (DLP) dynamics on  $\mathfrak{g}^*$  by  $\pi^{-1}$  corresponds to the image of the discrete Euler-Lagrange (DEL) dynamics on  $G \times G$  under the Legendre transformations  $F\mathbb{L}$  and results in an algorithm on  $T^*G$  approximating the continuous flow of the corresponding Hamiltonian system.*

*Idea of the proof.* The proof follows from the results of the previous section, in particular, diagram (3.2) relates the DLP dynamics on  $\mathfrak{g}^*$  with the DEP dynamics on  $\mathcal{U} \subset G$  which, in turn, is related to the DEL dynamics on  $\Delta \subset G \times G$  via the reconstruction.

**Theorem 4.2.** *The Poisson structure on the Lie group  $G$  obtained by reduction of the Lagrange symplectic form  $\omega_{\mathbb{L}}$  on  $\Delta \subset G \times G$  via  $\pi_d$  coincides with the Poisson structure on  $\mathcal{U} \subset G$  obtained by the pull-back of the Lie-Poisson structure  $\omega_{\mu}$  on  $\mathfrak{g}^*$  by the Legendre transformation  $F\ell$ . (see diagram (4.1) above).*

The proof is based on the commutativity of the following diagrams

$$\begin{array}{ccc} \Delta \subset G \times G & \xrightarrow{F\mathbb{L}} & T^*G \\ \downarrow \pi_d & & \downarrow \pi \\ \mathcal{U} \subset G & \xrightarrow{F\ell} & \mathfrak{g}^* \end{array} \qquad \begin{array}{ccc} \omega_{\mathbb{L}} & \xleftarrow{F\mathbb{L}^*} & \omega_{\text{can}} \\ & & \uparrow (\pi^{-1})^* \\ \omega_f & \xleftarrow{F\ell^*} & \omega_{\mu} \end{array}$$

and the  $G$  invariance of the unreduced symplectic forms.

## Discussions

**Main Results of this Paper.** We show that when a discrete Lagrangian  $\mathbb{L} : G \times G \rightarrow \mathbb{R}$  is  $G$ -invariant, a Poisson structure on (a subset) of one copy of the Lie group  $G$  can be defined which governs the corresponding discrete reduced dynamics. The symplectic leaves of this structure become dynamically invariant manifolds which are manifestly preserved under the structure preserving discrete Euler-Poincaré algorithm.

We apply Weinstein's results on Lagrangian mechanics on groupoids and algebroids [W 96] to the setting of regular Lie groups. Then, starting with a discrete Euler-Poincaré system on  $G$  one can readily recover, by means of the Legendre transformation, the corresponding Lie-Poisson Hamilton-Jacobi system on  $\mathfrak{g}^*$  analyzed by Ge and Marsden [GM 88].

**Various Important Remarks.** First of all, we remark that the discrete symplectic structure  $\omega_{\mathbb{L}}$  is not globally defined, but rather is only nondegenerate in a neighborhood  $\Delta$  of the diagonal in  $G \times G$ , i.e. whenever  $g_k$  and  $g_{k+1}$  are nearby. It follows then that the reduced Poisson structure  $\{f, h\}_G$  need only be defined on  $\mathcal{U}$ , where  $\mathcal{U}$  is the image of  $\Delta$ .

An important remark to Corollary 4.1 which follows from the results in [KMO 99] is that, in general, to get a corresponding algorithm on the Hamiltonian side which is consistent with the corresponding continuous Hamiltonian system on  $T^*G$ , one must use the time step  $h$ -dependent Legendre transform given by the map

$$(g_k, g_{k+1}) \mapsto (g_k, -hD_1\mathbb{L}(g_k, g_{k+1})).$$

The results of this paper are not effected, however, as we assume  $h$  to be constant and so we would simply add a constant multiplier to the corresponding symplectic and Poisson structures. For variable time-stepping algorithms, this remark is crucial and must be taken into account.

**More General Configuration Spaces or Where to Go.** The ideas outlined in this paper carry over to the integration of systems defined on a general configuration space  $M$  with some symmetry group  $G$ . In this case, the reduced discrete space  $(M \times M)/G$  inherits a Poisson structure from the one defined on  $M \times M$  (analogously to (4.5)). Its symplectic leaves again become dynamically invariant manifolds for structure-preserving integrators and can be viewed as images of the symplectic leaves of the reduced Poisson manifold  $T^*M/G$  under appropriately defined “Legendre transformations”.

The groupoid-algebroid formalism is very well suited to the discrete gauge field theory generalization as well as to discrete semi-direct product theory. The latter is related to the recent results of Bobenko and Suris [BS 98]. It would be very interesting to develop the semi-direct product point of view on the discrete level. The relation to Routhian reduction and how it can fit into the discrete semi-direct product theory should be further investigated.

Last but not least, the groupoid-algebroid formalism can be used to define a Poisson structure on a Lie algebra  $\mathfrak{g}$  using the duality between Lie-Poisson and Euler-Poincaré reduced systems on  $\mathfrak{g}^*$  and  $\mathfrak{g}$ , respectively. A reduced Lagrangian  $l$  determines the Legendre transformations  $Fl$  from  $\mathfrak{g}$  to  $\mathfrak{g}^*$  and its pull-back  $Fl^*$  defines Casimirs on  $\mathfrak{g}$  by  $C_{\mathfrak{g}}(\xi) = Fl^* \cdot C_{\mathfrak{g}^*}(\xi)$ . Besides purely theoretical interest, this can have applications for the analysis of dynamics on Lie algebras.

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## Appendix: Discrete Reduction

In this appendix we review the discrete Euler-Poincaré reduction of a Lagrangian system on  $G \times G$  considered in detail in [MPeS 98]. See [V 88, V 91, WM 97, MPS 98, LS 96, BS 98] for various related aspects of discrete mechanics. We approximate  $TG$  by  $G \times G$  and form a discrete Lagrangian  $\mathbb{L} : G \times G \rightarrow \mathbb{R}$  from the original Lagrangian  $L$  on  $TG$ . We choose discretization schemes for which the discrete Lagrangian  $\mathbb{L}$  inherits the symmetries of the original Lagrangian  $L$ :  $\mathbb{L}$  is  $G$ -invariant on  $G \times G$  whenever  $L$  is  $G$ -invariant on  $TG$ . In particular, the induced right (left) lifted action of  $G$  onto  $TG$  corresponds to the diagonal right (left) action of  $G$  on  $G \times G$ . Then, application of discrete variational principle results in discrete Euler-Lagrange (DEL) equations as well as the discrete symplectic form  $\omega_{\mathbb{L}}$ .

**The discrete Euler-Poincaré algorithm.** The discrete reduction of a right-invariant system proceeds as follows. The induced group action on  $G \times G$  by an element  $\bar{g} \in G$  is simply right multiplication in each component:  $\bar{g} : (g_k, g_{k+1}) \mapsto (g_k \bar{g}, g_{k+1} \bar{g})$ , for all  $g_k, g_{k+1} \in G$ . (Of course, some systems such as the rigid body are left invariant.)

The quotient map is given by  $\pi_d : G \times G \rightarrow (G \times G)/G \cong G$ ,  $(g_k, g_{k+1}) \mapsto g_k g_{k+1}^{-1}$ . We note that one may alternatively use  $g_{k+1} g_k^{-1}$  instead of  $g_k g_{k+1}^{-1}$  as the quotient map; this alternative choice is used in [BS 98]. The projection map defines the **reduced discrete Lagrangian**  $\ell : G \rightarrow \mathbb{R}$  for any  $G$ -invariant  $\mathbb{L}$  by  $\ell \circ \pi_d = \mathbb{L}$ , so that  $\ell(g_k g_{k+1}^{-1}) = \mathbb{L}(g_k, g_{k+1})$ .

A reduced discrete variational principle results in the **discrete Euler-Poincaré** (DEP) equations

$$R_{f_{kk+1}}^* \ell'(f_{kk+1}) - L_{f_{k-1k}}^* \ell'(f_{k-1k}) = 0 \quad (4.3)$$

for  $k = 1, \dots, N-1$ , where  $R_f^*$  and  $L_f^*$  are the right and left pull-backs by  $f$ , respectively, and  $\ell' : G \rightarrow T^*G$  is the differential of  $\ell$  defined as follows. Let  $g^\epsilon$  be a smooth curve in  $G$  such that  $g^0 = g$  and  $(d/d\epsilon)|_{\epsilon=0} g^\epsilon = v$ , then  $\ell'(g) \cdot v = (d/d\epsilon)|_{\epsilon=0} \ell(g^\epsilon)$ . In the case that  $\mathbb{L}$  is left invariant, the discrete Euler-Poincaré equations take the form

$$L_{f_{k+1k}}^* \ell'(f_{k+1k}) - R_{f_{kk-1}}^* \ell'(f_{kk-1}) = 0 \quad (4.4)$$

where  $f_{k+1k} \equiv g_{k+1}^{-1} g_k$  is in the left quotient  $(G \times G)/G$ .

The symplectic structure  $\omega_{\mathbb{L}}$  naturally defines a Poisson structure on  $\Delta \subset G \times G$  (which we shall denote  $\{\cdot, \cdot\}_{G \times G}$ ) by the relation  $\{F, H\}_{G \times G} = \omega_{\mathbb{L}}(X_F, X_H)$ . Then, Theorem 2.2 of [MPeS 98] states that if the action of  $G$  on  $G \times G$  is proper, the algorithm on  $G$  defined by the discrete Euler-Poincaré equations (4.3) preserves the induced Poisson structure  $\{\cdot, \cdot\}_G$  on  $\mathcal{U} \subset G$  given by

$$\{f, h\}_G \circ \pi_d = \{f \circ \pi_d, h \circ \pi_d\}_{G \times G} \quad (4.5)$$

for any  $C^1$  functions  $f, h$  on  $\mathcal{U}$ , where  $\mathcal{U} = \pi_d(\Delta)$ .

**Reconstruction.** Using the definition  $f_{kk+1} = g_k g_{k+1}^{-1}$ , the DEL algorithm can be reconstructed from the DEP algorithm by

$$(g_{k-1}, g_k) \mapsto (g_k, g_{k+1}) = (f_{k-1k}^{-1} \cdot g_{k-1}, f_{kk+1}^{-1} \cdot g_k),$$

where  $f_{kk+1}$  is the solution of (4.3). Indeed,  $f_{kk+1}^{-1} \cdot g_k$  is precisely  $g_{k+1}$ . Similarly one shows that in the case of a left  $G$  action, the reconstruction of the DEP equations (4.4) is given by  $(g_{k-1}, g_k) \mapsto (g_k, g_{k+1}) = (g_{k-1} \cdot f_{kk-1}^{-1}, g_k \cdot f_{k+1k}^{-1})$ .

**The discrete Lie-Poisson algorithm** In addition to reconstructing the dynamics on  $\Delta \subset G \times G$ , one may use the coadjoint action to form a **discrete Lie-Poisson** algorithm approximating the dynamics on  $\mathfrak{g}^*$  [MPeS 98]

$$\mu_{k+1} = \text{Ad}_{f_{kk+1}}^* \cdot \mu_k, \quad (4.6)$$

where  $\mu_k := \text{Ad}_{g_k}^* \mu_0$ ,  $\mu_0$  is the constant of motion (the momentum map value), and the sequence  $\{f_{kk+1}\}$  is provided by the DEP algorithm on  $G$ . The corresponding discrete Lie-Poisson equations for the left invariant system is given by  $\Pi_{k+1} = \text{Ad}_{f_{k+1k}}^* \cdot \Pi_k$ , where  $\Pi_k := \text{Ad}_{g_k}^* \pi_0$  and  $\pi_0$  is the constant momentum map.

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